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Temperature Evolution During Radiative Gravitational Collapse

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We investigate the evolution of the temperature profile of a Friedmann-like collapsing sphere undergoing dissipative gravitational collapse in the form of a radial heat flux. We further consider the behavior of the star close to quasi-static equilibrium (weak heat flux approximation) and show that relaxational effects cannot be ignored. It is explicitly shown that extended irreversible thermodynamics predict a higher temperature at all interior points of the stellar configuration compared to the Eckart theory. These results carry over to the weak heat flux approximation with the magnitude of the temperature being lower than the full radiating model. The stability of the model after its departure from equilibrium is studied by considering the behavior of the "control parameter" throughout the stellar interior.

KEY WORDS: gravitational collapse; heat flow; thermodynamics.

1. INTRODUCTION

The problem of gravitational collapse has many interesting applications in astrophysics where formation of compact stellar objects such as white dwarfs and neutron stars are usually preceded by a period of radiative collapse. The simplest scenario is the static case. However, the study of static spheres is an idealised problem since astronomical observations indicate that most, if not all, gravitating systems are nonstatic. Additionally, radiative processes are vital mechanisms of energy dissipation in such systems. The surface of a collapsing star divides spacetime into two distinct regions; the interior region and the exterior region. Since the star is radiating the exterior spacetime is no longer described by the exterior Schwarzschild solution but is now represented by the Vaidya solution for pure outgoing radiation. The interior matter distribution is described by a spherically symmetric, shear-free line element for a generalised energy–momentum tensor

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with heat flow. By utilising Raychaudhuri's equation we can show that the slowest possible collapse is for shear-free matter distributions (Bonnor *et al.*, 1989). The interior spacetime has to be matched to the exterior spacetime at the boundary of the radiating star. Hence, to obtain a complete picture of the gravitational collapse of a star it is necessary to adequately describe the interior and exterior spacetimes and to provide the matching conditions for them.

The junction conditions for a spherically symmetric radiating star was first derived by Santos (1985). He was able to show, with the use of the junction conditions, that the pressure on the boundary of a radiating sphere cannot vanish. This important result has since become a crucial requirement for spherically symmetric, shear-free radiative collapse. Several physically reasonable models of radiative spherical collapse with heat flow have been proposed by utilising the junction conditions derived by Santos. A common shortfall of these early investigations was the lack of thermodynamical consideration of the stellar fluid. Little or no attempt was made to investigate the evolution of the temperature profile during the collapse. Also, early attempts to determine the behavior of the temperature in these models were carried out within the framework of the Eckart formalism (Grammenos, 1994). It was assumed that the fluid was close to equilibrium at all times and that relaxational effects were neglible. Several investigations using extended irreversible thermodynamics showed that causal thermodynamics predict significantly different results from its noncausal counterpart (Di Prisco *et al.*, 1996; Govender *et al.*, 1998, 1999; Herrera *et al.*, 1997).

In this paper we consider a simple stellar model in which the fluid trajectories within the stellar core are geodesics. This model was first presented by Kolassis *et al.* (1988) who considered a Friedmann-like interior matched to the outgoing Vaidya solution. Subsequent work by Chan *et al.* (1989) extended this model to include the weak heat flux approximation where calculations showed that the pressure gradient changes sign leading to increased instability of the collapsing star. In the present investigation we look at the evolution of the temperature in the weak heat flux approximation. Furthermore, we provide new solutions in which the collision time is not constant during the collapse process. Utilising graphical plots we highlight differences in the temperature profile in the full Friedmann-like model and the weak heat flux approximation. We further analyze the behavior of the model by considering the evolution of a "control parameter" dictated by causality requirements.

2. RADIATING COLLAPSE MODEL

For the Friedmann-like radiating model the interior matter distribution is described by the spherically symmetric, shear-free metric

$$ds^2 = -dt^2 + A(t,r)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{1}$$

$$A = \frac{M}{2b}\left[\frac{1-b^2\lambda(t)}{1-r^2\lambda(t)}\right]u(t)^2, \tag{2}$$

where $u = (6t/M)^{1/3}$, $\lambda = a\exp u$, and a, b, and M are constants. The fluid four-velocity is $u^\alpha = \delta^\alpha_0$ and the fluid volume collapse rate is determined from

$$\Theta = 3\frac{\dot{A}}{A}. \tag{3}$$

We note that the four-acceleration and shear are simultaneously vanishing.

The heat flux (which is the total energy flux, since there is no particle flux relative to u^α) has the form

$$q_\alpha = q(t, r)n_\alpha,$$

where n_α is a unit radial vector, so that q is a covariant scalar measure of the heat flux ($q^2 = q^\alpha q_\alpha$). The other dynamical covariant scalars are the energy density ρ and isotropic pressure p. Following Grammenos (1994) the Einstein field equations yield (using units with $c = 1 = 8\pi G$)

$$\rho = \frac{12}{M^2u^4}\left\{\left[\frac{2}{u} - \frac{(b^2-r^2)\lambda}{(1-b^2\lambda)(1-r^2\lambda)}\right]^2 - \frac{4b^2\lambda}{(1-b^2\lambda)^2}\right\}, \tag{4}$$

$$p = \frac{4}{M^2u^4}\frac{(b^2-r^2)\lambda}{(1-b^2\lambda)(1-r^2\lambda)}\left[\frac{8}{u} + \frac{5}{1-r^2\lambda} - \frac{1}{1-b^2\lambda} - 2\right] + \frac{16}{M^2u^4}\frac{b^2\lambda}{(1-b^2\lambda)^2}, \tag{5}$$

$$q = \frac{16br\lambda}{M^2u^4(1-b^2\lambda)(1-r^2\lambda)}. \tag{6}$$

Since the star is radiating energy, the exterior spacetime is described by the Vaidya metric given by

$$ds^2 = -\left[1 - \frac{2m(v)}{R}\right]dv^2 - 2\,dv\,dR + R^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{7}$$

where m represents the Newtonian mass of the gravitating body as measured by an observer at infinity. The smooth matching of the interior metric (1) to the Vaidya metric (7) fixes the temporal evolution of our model.

The physical properties of this model have been studied extensively in the past (Bonnor *et al.*, 1989) and the model has been shown to be reasonably well-behaved. The collapse starts at $u = -\infty$ with an infinite radius and zero density, and evolves to $u = u_{\mathrm{H}}$, the time of formation of the horizon. As pointed out in Govender *et al.* (1998) the hydrodynamical behavior of the stellar interior is the same as in the corresponding noncausal model.

3. CAUSAL HEAT TRANSPORT

In this section we consider the temperature profile of our model by adopting the relativistic thermodynamics of Israel and Stewart (1979). As we pointed out earlier, the Eckart theory leads to a parabolic diffusion equation that predicts infinite propagation velocities for the the dissipative fluxes. Furthermore, all the equilibrium states are unstable. The causal heat transport equation with no viscous/heat and vorticity/heat couplings is given by

$$\tau h_{\alpha\beta}\dot{q}_{\beta} + q_{\alpha} = -\kappa(h_{\alpha\beta}\nabla_{\beta}T + T\dot{u}_{\alpha}), \tag{8}$$

where $h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}u_{\beta}$ projects into the comoving rest space, $g_{\alpha\beta}$ is the metric, T is the local equilibrium temperature, $\kappa(\geq 0)$ is the thermal conductivity, and $\tau(\geq 0)$ is the relaxational time. The relaxation time can be considered as the time taken by the corresponding dissipative flux to attain a steady value (Anile *et al.*, 1998).

The Maxwell–Fourier law is regained by setting $\tau = 0$ in (8). The origin of the noncausal nature of the Maxwell–Fourier law arises from the fact that the appearance of a temperature gradient results in an instantaneous heat flux.

In order to solve the causal heat transport equation (8), we follow the arguments set out in (Govender *et al.*, 1998) in which it is assumed that neutrinos generated in the stellar core are responsible for heat dissipation to the exterior. The thermal conductivity has the form

$$\kappa = \gamma T^3 \tau_{\mathrm{c}}, \tag{9}$$

where γ (≥ 0) is a constant and τ_{c} is the mean collision time between massless and massive particles. Assuming that

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right)\tau_{\mathrm{c}} = \beta T^{-\sigma}, \tag{10}$$

where α (≥ 0), β (≥ 0), and σ (≥ 0) are constants, and taking the metric (1) into account, (8) becomes

$$\alpha T^{3-\sigma}\frac{dT}{ds} + \beta\left(\dot{f} + \frac{1}{2}s\right)T^{-\sigma} + 1 = 0, \tag{11}$$

where

$$s = \frac{4}{Mu^2(1 - r^2\lambda)},$$

$$f(t) = -\ln[u^2(1 - b^2\lambda)].$$

The function of integration that arises in the solution of (11) is fixed by the effective surface temperature of a star given by (Di Prisco *et al.*, 1996)

$$(T^4)_\Sigma = \left(\frac{L}{4\pi\delta r^2 A^2}\right)_\Sigma, \tag{12}$$

where $\delta\,(>0)$ is a constant and the total luminosity at ∞, L, is

$$L = -\frac{dm}{dv} = \frac{2b^2\lambda}{(1-b^2\lambda)^2}\left[\frac{2}{u} + \frac{(1+b^2\lambda)}{(1-b^2\lambda)}\right]^2. \tag{13}$$

The noncausal temperature $\tilde{T}$ is obtained by setting $\beta = 0$ in (11) and integrating (Govinder and Govender, 2001):

$$\begin{aligned}\sigma \neq 4:\quad T^{4-\sigma} &= \frac{4\lambda(4-\sigma)(b^2-r^2)}{\alpha M u^2(1-r^2\lambda)(1-b^2\lambda)}\\ &\quad + \left\{\frac{2b^2\lambda}{\pi\delta M^2u^4(1-b^2\lambda)^2}\left[\frac{2}{u} + \left(\frac{1+b^2\lambda}{1-b^2\lambda}\right)\right]^2\right\}^{1-\sigma/4}, \end{aligned}\tag{14}$$

$$\begin{aligned}\sigma = 4:\quad T^4 &= \exp\left[\frac{16\lambda(b^2-r^2)}{\alpha M u^2(1-r^2\lambda)(1-b^2\lambda)}\right]\\ &\quad \times \left\{\frac{2b^2\lambda}{\pi\delta M^2u^4(1-b^2\lambda)^2}\left[\frac{2}{u} + \left(\frac{1+b^2\lambda}{1-b^2\lambda}\right)\right]^2\right\}. \end{aligned}\tag{15}$$

We can find three causal solutions for (8) (Govinder and Govender, 2001). When $\sigma = 0$ (i.e., for constant mean collision time) we have

$$\begin{aligned}T^4 &= \frac{16\lambda(b^2-r^2)}{\alpha M u^2(1-r^2\lambda)(1-b^2\lambda)}\left\{\frac{\beta[2-(b^2+r^2)\lambda]}{Mu^2(1-b^2\lambda)(1-r^2\lambda)}\right.\\ &\quad \left. + \beta\dot{f} + 1\right\} + \frac{2b^2\lambda}{\pi\delta M^2u^4(1-b^2\lambda)^2}\left[\frac{2}{u} + \left(\frac{1+b^2\lambda}{1-b^2\lambda}\right)\right]^2, \end{aligned}\tag{16}$$

and for $\sigma = 4$ we have

$$\begin{aligned}T^4 &= \exp\left[\frac{16\lambda(b^2-r^2)}{\alpha M u^2(1-r^2\lambda)(1-b^2\lambda)}\right]\\ &\quad \times \left\{\frac{2b^2\lambda}{\pi\delta M^2u^4(1-b^2\lambda)^2}\left[\frac{2}{u} + \left(\frac{1+b^2\lambda}{1-b^2\lambda}\right)\right]^2\right\}\\ &\quad + \frac{\beta}{8}\left\{\alpha - 8\dot{f} - \frac{16}{Mu^2(1-r^2\lambda)} - \left[\alpha - 8\dot{f} - \frac{16}{Mu^2(1-b^2\lambda)}\right]\right.\\ &\quad \left. \times \exp\left[\frac{16\lambda(b^2-r^2)}{\alpha M u^2(1-r^2\lambda)(1-b^2\lambda)}\right]\right\}. \end{aligned}\tag{17}$$

For $\sigma = 2$ one must consider three cases depending on the relationship between α and β. However, the solutions there are not amenable to further manipulation (Govinder and Govender, 2001).

4. WEAK HEAT FLUX APPROXIMATION

4.1. The Field Equations

In this section we consider the case where the heat flux q is small, which could be the case when the fluid is close to equilibrium. This approximation is equivalent to assuming

$$0 < a \ll 1, \tag{18}$$

where a appears in $\lambda = a \exp u$. The Einstein field equations (4)–(6) reduce to (Chan *et al.*, 1989)

$$\rho \approx \frac{48}{M^2u^6}\{1 - \lambda[(b^2 - r^2)u + b^2u^2]\}, \tag{19}$$

$$p \approx \frac{8\lambda}{M^2u^4}\left[\left(\frac{4}{u} + 1\right)(b^2 - r^2) + 2b^2\right], \tag{20}$$

$$q \approx \frac{16br\lambda}{M^2u^4}. \tag{21}$$

The star starts collapsing at $u = -\infty$ until the time of formation of the horizon, at $u_{\mathrm{H}} \approx -2 + 4b^2\lambda_{\mathrm{H}}$ (Chan *et al.*, 1989). It can be shown that the pressure remains positive throughout the stellar interior until the time of formation of horizon. The gradient of the pressure can be obtained from (20):

$$p' \approx -\frac{16\lambda}{M^2u^4}\left(\frac{4}{u} + 1\right)r,$$

and can change sign during the collapse process. The calculation of the effective adiabatic index shows that the central regions are more dynamically unstable than the surface layers. For our weak heat flux model the causal heat transport equation (8) reduces to

$$\alpha T^{3-\sigma}\frac{dT}{ds} + \beta(\dot{f} + \frac{1}{2}s)T^{-\sigma} + 1 = 0, \tag{22}$$

where

$$s = \frac{4\lambda r^2}{Mu^2},$$

$$f(t) = -2\,\ln u.$$

The function of integration that arises in the solution of (22) is fixed by the effective surface temperature of a star given by (12) where the total luminosity at ∞, L, is now given by

$$L \approx 2b^2\lambda\left[1+\frac{2}{u}\right]^2. \tag{23}$$

4.2. Exact Solutions of the Temperature Equation

In the noncausal case ($\beta = 0$) we find the general solution of (22) to be

$$\sigma \neq 4: \quad T^{4-\sigma} = \frac{4(4-\sigma)\lambda}{\alpha M u^2}(b^2 - r^2) + \left[\frac{2b^2\lambda}{\pi\delta M^2 u^4}\left(1+\frac{2}{u}\right)^2\right]^{(4-\sigma)/4}, \tag{24}$$

$$\sigma = 4: \quad T^4 = \exp\left[\frac{16\lambda}{\alpha M u^2}(b^2 - r^2)\right]\left[\frac{2b^2\lambda}{\pi\delta M^2 u^4}\left(1+\frac{2}{u}\right)^2\right]. \tag{25}$$

For nonzero β (i.e., the causal case) it is difficult to find the general analytic solution. In the case of constant mean collision time ($\sigma = 0$) we obtain

$$T^4 = \frac{4\beta}{\alpha}\left[\frac{4\dot{f}(b^2-r^2)\lambda}{Mu^2} + \frac{4(b^4-r^4)\lambda^2}{M^2u^4}\right] + \frac{16(b^2-r^2)\lambda}{\alpha M u^2} + \frac{2b^2\lambda}{\pi\delta M^2 u^4}\left(1+\frac{2}{u}\right)^2, \tag{26}$$

while the only meaningful variable mean collision time solution we have been able to find is in the case of $\sigma = 4$:

$$T^4 = \exp\left[\frac{16\lambda(b^2-r^2)}{\alpha M u^2}\right]\left[\frac{2b^2\lambda}{\pi\delta M^2 u^4}\left(1+\frac{2}{u}\right)^2 + \beta\dot{f} - 2\beta\left(\frac{\alpha}{16} - \frac{\lambda b^2}{Mu^2}\right)\right] - \beta\dot{f} + 2\beta\left(\frac{\alpha}{16} - \frac{\lambda r^2}{Mu^2}\right). \tag{27}$$

(As before, we note that a solution does exist for $\sigma = 2$ but that solution does not lend itself to a simple analysis of the temperature profile.)

It is clear from Figs. 1–4 that the casual temperature is everywhere greater than the noncausal temperature, the difference being much greater in the full radiative case. This is to be expected since the energy output in the full radiative case is much greater than the weak heat flux limit. One could understand this as the collapse proceeding sufficiently far enough so as to produce vast amounts of neutrinos that

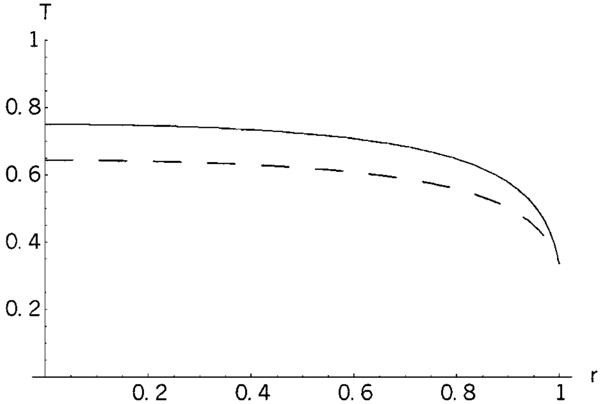

Fig. 1. Plots of the causal (solid) and noncausal (dotted) temperature (T) profiles in the full radiative case for constant mean collison time against radius (r) with time constant ($u = -1$, $\beta = \delta = b = 1$, $\alpha = M = 10$, $\lambda = 0.5$).

are responsible for heat generation within the stellar in the full case. The relative radial gradient of the causal temperature is everywhere greater than the noncausal temperature gradient with the difference being more marked at the surface of the star.

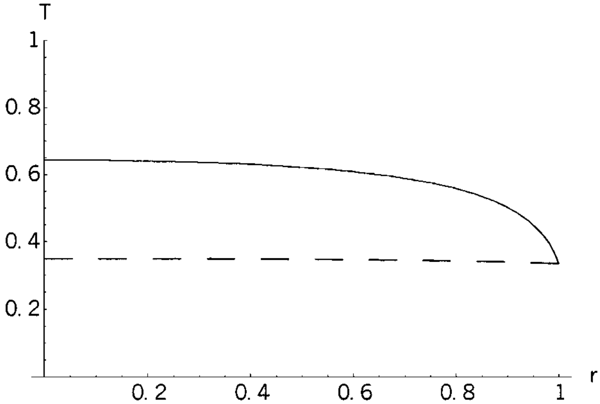

Fig. 2. Plots of the causal (solid) and noncausal (dotted) temperature (T) profiles in the full radiative case for variable mean collison time ($\sigma = 4$) against radius (r) with time constant ($u = -1$, $\beta = \delta = b = 1$, $\alpha = M = 10$, $\lambda = 0.5$).

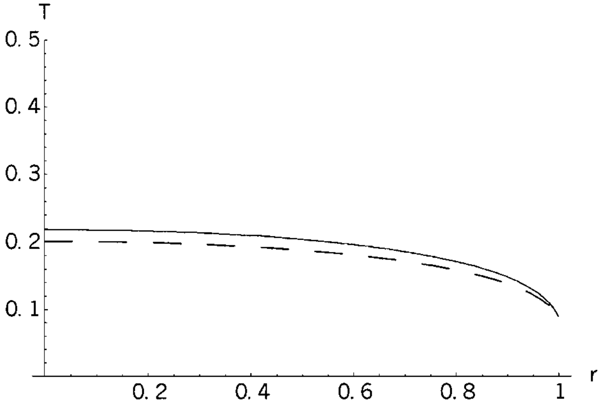

Fig. 3. Plots of the causal (solid) and noncausal (dotted) temperature (T) profiles in the weak heat flux limit for constant mean collison time against radius (r) with time constant ($u = -1$, $\alpha = \delta = b = 1$, $\beta = M = 10$, $\lambda = 0.009895$).

4.3. Stability Requirements

The evolution of a radiating relativistic star close to quasi-static equilibrium has recently received widespread attention (Herrera *et al.*, 1997; Herrera and Martinez, 1997, 1998). It has been shown that the subsequent evolution of the star after its departure from equilibrium is sensitive to a certain "control parameter,"

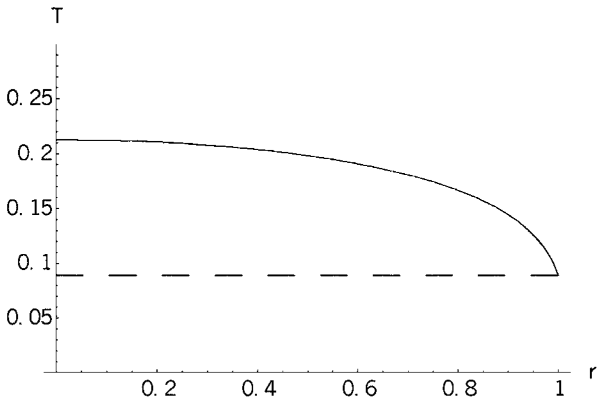

Fig. 4. Plots of the causal (solid) and noncausal (dotted) temperature (T) profiles in the weak heat flux limit for variable mean collison ($\sigma = 4$) against radius (r) with time constant ($u = -1$, $\alpha = \delta = b = 1$, $\beta = M = 10$, $\lambda = 0.009895$).

which, in the absence of bulk viscosity and shear viscosity, is given by

$$\alpha_1 = \frac{1}{(\rho + p)}\left(\frac{\kappa T}{\tau}\right),$$

where τ is the relaxation time. More recently it has been reinforced that the control parameter α_1 measures the instability of the system (Herrera *et al.*, 2000). The condition $\alpha_1 = 1$ is called the critical point and implies vanishing of the effective inertial mass density of a fluid element. The effective inertial mass density decreases as α_1 increases from zero, thus signifying the onset of instability. It has been further shown that causality requirements demand that $\alpha_1 < 1$. For our model the above parameter reduces to

$$\alpha_1 = \frac{\alpha M^2 u^6 T^4}{8\beta[6 - 2\lambda u(b^2 - r^2) - \lambda u^2(3b^2 + r^2)]}.$$

Plots of the effective adiabatic index of our model show that central regions of the stellar configuration are more dynamically unstable than the outer regions (Chan *et al.*, 1989). From Figs. 5 and 6 we note that α_1 is maxiumum at the center and diminishes outwards towards the boundary in keeping with the numerical results found in Chan *et al.* (1989). Furthermore, we should point out that α_1 for the variable collision time is lower than that for the constant collision time, thus signifying greater stability for a physically motivated choice of the collision time. The choice of a constant collision time may only be valid for a very short period and cannot be used throughout the collapse. Also, note that $\alpha_1 < 1$ throughout the stellar interior thus implying that the critical point is never reached provided that the fluid is always close to equilibrium.

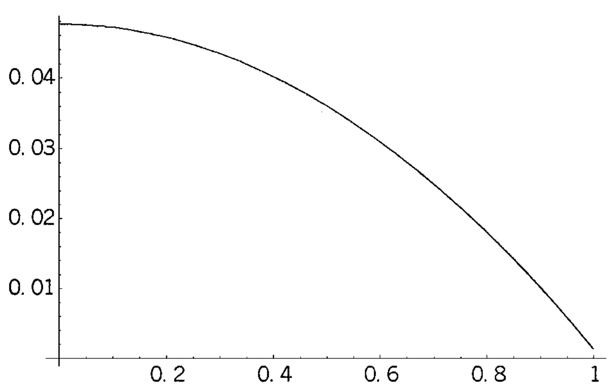

Fig. 5. Plot of the “control parameter” against radius r (causal temperature profile with constant mean collison time) with time constant ($u = -1$, $\alpha = \delta = b = 1$, $\beta = M = 10$, $\lambda = 0.009895$).

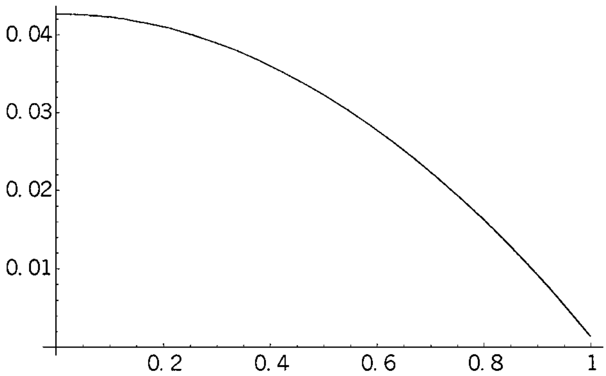

Fig. 6. Plot of the "control parameter" against radius r (causal temperature profile with variable mean collison time ($\sigma = 4$)) with time constant ($u = -1$, $\alpha = \delta = b = 1$, $\beta = M = 10$, $\lambda = 0.009895$).

5. CONCLUSION

We have fully analysed the evolution of the temperature profile within the framework of extended irreversible thermodynamics in a simple stellar model undergoing dissipative gravitational collapse. We have further considered the weak heat flux approximation where it is assumed that the star is always close to quasi-static equilibrium. The causal heat transport equation is solved for variable collision times and the solutions reported here for the weak heat flux approximation are new. The temperature profiles strongly indicate that relaxational effects, especially when the fluid is far from equilibrium, cannot be ignored. An analysis of the stability of the model confirms earlier findings that the control parameter α_1 is a measure of the stability of a system, at least before the critical point is reached. It would be interesting to investigate the effect of shear and heat flow during dissipative gravitational collapse and to ascertain the predicting power of the control parameter on the stability of the star.

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